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The BCS model and the off-shell Bethe ansatz for vertex models

Luigi Amico[1,2], **G Falci**[1,2,3] **and Rosario Fazio**[1,2]

[1] Dipartimento di Metodologie Fisiche e Chimiche (DMFCI), Universitá di Catania, viale A. Doria 6, I-95125 Catania, Italy
[2] Istituto Nazionale per la Fisica della Materia, Unitá di Catania, Italy
[3] Laboratoire d'Etudes des Propriétés Electroniques des Solides, Centre National de la Recherche Scientifique, BP 166, 38042 Grenoble Cedex 9, Grenoble, France

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Abstract
We study the connection between the BCS pairing model and the inhomogeneous vertex model. The two spectral problems coincide in the quasi-classical limit of the off-shell Bethe ansatz of the disordered six-vertex model. The latter problem is transformed into an auxiliary spectral problem which corresponds to the diagonalization of the integrals of motion of the BCS model. A generating functional whose quasi-classical expansion leads to the constants of motion of the BCS model, and in particular the Hamiltonian, is identified.

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1. Introduction

One of the most successful models of interacting electrons is the BCS model of pairing [1]. Originally proposed to describe properties of superconductors [2], the pairing idea has been applied to a large variety of physical systems in nuclear physics [3] and in QCD [4]. Recent experiments in metallic nanoparticles [5] have renewed the interest in the problem of pairing correlations in mesoscopic systems [6]. The BCS Hamiltonian consists of a kinetic and an interaction term which describes the attraction between electrons in time-reversed states

$$H = \sum_{\substack{j=1\\ \sigma=\uparrow,\downarrow}}^{\Omega} \varepsilon_{j\sigma} c^{\dagger}_{j\sigma} c_{j\sigma} - g \sum_{j,j'=1}^{\Omega} c^{\dagger}_{j\uparrow} c^{\dagger}_{j\downarrow} c_{j'\downarrow} c_{j'\uparrow}. \tag{1}$$

The quantum number $j \in \{1, \ldots, \Omega\}$, $\sigma \in \{\uparrow, \downarrow\}$ labels a shell of doubly degenerate time-reversed states of energy ϵ_j; $c_{j,\sigma}$ and $c^{\dagger}_{j,\sigma}$ are the corresponding electronic operators, g is the BCS coupling constant. The low-energy properties associated to this model are universal functions of the ratio between two energies, the single-particle average level spacing and the BCS gap [6].

Various exact results have been obtained for the BCS Hamiltonian.

In the limit $g \to \infty$ the exact eigenvalues and eigenstates can be found (see, for example, [7]) and the integrals of motion are Gaudin Hamiltonians [8]. An important consequence of the relation with the isotropic Gaudin magnet (discussed in appendix A), is that the quantum inverse scattering method (QISM) [9] for the $g \to \infty$ BCS model can be related with the QISM for the Gaudin model [10]. (The same set of operators already emerged from the quasiclassical expansion of the *twisted* monodromy matrix of the inhomogeneous vertex model.) The $g \to \infty$ BCS model can be also related to the inhomogeneous vertex models [11]. (The inhomogeneous vertex models are related to the Gaudin models [10, 15]: the BA equations of the Gaudin model can be obtained by taking the quasi-classical limit of the BA equations of vertex models.)

Much less work has been done for finite g. The exact solution was found by Richardson and Sherman (RS) [16] and independently by Gaudin [17] by means of the Bethe ansatz (BA) technique. Approximate expression of correlation functions were found in [18]. More recently, the integrals of motion of the BCS model were obtained [10, 19] and were diagonalized by means of the algebraic BA.

In this paper we show that for finite g the BCS model is connected to a disordered six-vertex model through the *off-shell* BA (OSBA) introduced by Babujian *et al* in [15]. (The OSBA deals with the off-diagonal terms generated by the application of the transfer matrix to the Bethe vectors [15].) In this framework, the known connection between the isotropic Gaudin models and the inhomogeneous vertex model is obtained as the *mass shell* limit which corresponds to $g \to \infty$.

A strong hint towards our result is provided by a recent work by Sierra [20] who has shown connection between the BCS pairing model and a $su(2)_c$ Wess–Zumino–Novikov–Witten conformal field theory (CFT), in the singular limit when the central charge is infinite; the RS wavefunctions solve the Knizhnik–Zamolodchikov equations for the CFT correlation functions. The results of Sierra are indeed related to the connection existing between models in statistical mechanics and correlation functions of a suitable CFT established through the OSBA. In fact the solution of the quasi-classical OSBA equations is equivalent to solution of the Knizhnik–Zamolodchikov equations [21, 22]. In particular, the quasi-classical OSBA equations for the vertex models generate the correlators of the $su(2)$ Wess–Zumino–Novikov–Witten CFT.

The paper is organized as follows. In section 2 we review the exact solution of the pairing model. In section 3 the quasi-classical expansion of the OSBA of the disordered six-vertex model is identified as the diagonalization of the BCS model. Section 4 is devoted to the conclusions. The connection between the diagonalization of the pairing model for infinite pairing coupling constant g and the diagonalization of Gaudin magnet is reviewed in appendix A. In appendix B we summarize the QISM of the inhomogeneous vertex model.

2. The exact solution of the pairing Hamiltonian

In this section we review the exact solution [16, 17] of the BCS model (equation (1)) and the formulation of its integrability. Due to the form of the pairing interaction in equation (1), single occupied states are frozen and we can focus on scattering of pairs. The Schrödinger equation for a state of N Cooper pairs

$$H|N\rangle = \mathcal{E}|N\rangle \tag{2}$$

has the solution [16, 17]

$$|N\rangle = \prod_{\alpha=1}^{N} \sigma^+(e_\alpha, \varepsilon)|0\rangle \qquad \sigma^+(e_\alpha, \varepsilon) \ := \ \sum_{j=1}^{\Omega} \frac{\sigma_j^\dagger}{2\varepsilon_j - e_\alpha}$$
$$\mathcal{E} = \sum_{\alpha=1}^{N} e_\alpha. \tag{3}$$

The operators $\sigma_j^- := c_{j,\downarrow}c_{j,\uparrow}$, $\sigma_j^+ = (\sigma_j^-)^\dagger$ and $\sigma_j^z := (c_{j,\uparrow}^\dagger c_{j,\uparrow} + c_{j,\downarrow}^\dagger c_{j,\downarrow} - 1)/2$ realize $su(2)$ in the lowest representation. The vacuum state is the highest weight vector of $su(2)$: $|0\rangle := |1/2, -1/2\rangle$. The operators $\{\sigma^\pm(e_\alpha, \varepsilon), \sigma^z(e_\alpha, \varepsilon)\}$ generate (for generic e_α) the Gaudin algebra $\mathcal{G}[sl(2)]$ (see appendix A and equation (15) of the next section). The energy $\mathcal{E}$ is given in terms of the spectral parameters e_α which satisfy the algebraic equation [16]

$$\frac{1}{g} + \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{2}{e_\beta - e_\alpha} - \sum_{j=1}^{\Omega} \frac{1}{2\varepsilon_j - e_\alpha} = 0 \qquad \alpha = 1, \ldots, N. \tag{4}$$

The method employed by RS has analogies with the coordinate BA technique. In fact, in the coordinate BA the ansatz functions are plane waves (describing free particles) modified to include the interaction. In the RS solution the ansatz functions are the solutions of the model when pairs of time-reversed electrons are treated as bosons; these functions are modified because Cooper pairs behave as hard-core bosons. In both the RS and the BA procedures the modification enters the set of the algebraic equations for the rapidities (Bethe equations) parametrizing the eigenvalues of the Hamiltonian.

By using the spin realization of pair operators $\{\sigma_j^z, \sigma_j^\pm\}$, the pairing Hamiltonian can be written as a quantum spin model with long-range interaction in a non-uniform fictitious magnetic field, given by ε_j

$$H = \sum_{j=1}^{\Omega} \varepsilon_j \sigma_j^z - \frac{g}{2} \sum_{j,l=1}^{\Omega} (\sigma_l^+ \, \sigma_j^- + \sigma_j^+ \, \sigma_l^-) + \text{const}. \tag{5}$$

Cambiaggio *et al* [19] found that the integrals of motion τ_j of this model, if $\varepsilon_j \neq \varepsilon_l, \forall j \neq l$, have the form

$$\tau_j \ = \ \frac{1}{g}\sigma_j^z - \ \Xi_j \tag{6}$$

and satisfy the commutation relations $[H, \tau_j] = [\tau_j, \tau_l] = 0$, $\forall j, l \in \{1, \ldots, \Omega\}$. The operators Ξ_j in (6) are spin-$\frac{1}{2}$ Gaudin Hamiltonians [8]

$$\Xi_j \ := \ \sum_{\substack{l=1\\ l\neq j}}^{\Omega} \frac{\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_l}{\varepsilon_j - \varepsilon_l}. \tag{7}$$

The commuting operators τ_j were also found by Sklyanin [10] by taking the quasi-classical limit of the monodromy matrix of the inhomogeneous vertex model *twisted* by a term proportional to σ_j^z/g. The pairing Hamiltonian can be expressed as function of the integrals of motion as

$$\frac{1}{g^3} H \ = \ \frac{1}{g^2} \sum_{j=1}^{\Omega} \varepsilon_j \, \tau_j \ + \ \sum_{j,l=1}^{\Omega} \tau_j \, \tau_l \ + \text{const}. \tag{8}$$

In the limit $g \to \infty$ the problem is equivalent to the diagonalization of (all) the Gaudin Hamiltonians (see appendix A for details).

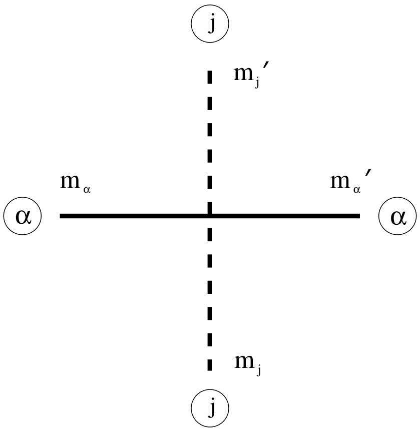

Figure 1. The vertex $(\alpha, j) \in \Lambda$. Disorder z_j is distributed along the vertical lines. It is due to both the spin inhomogeneity and impurities I_j distributed in the lattice: $z_j \equiv z(S_j) + z(I_j)$.

3. The OSBA of the inhomogeneous vertex model and the pairing model

Vertex models are two-dimensional classical models which were solved long ago by inverse methods [11]. Generalizations to $su(2)$ higher representations and to include disorder were intensively studied [12–14].

In this section we introduce a vertex model in which the inhomogeneity is due to the combination of given (see below) disordered distribution of both the spin and the impurities in the lattice. Then we apply the scheme developed by Babujian *et al* [15] to relate this inhomogeneous vertex model to the BCS model equation (1).

The model is defined in the following way. On the edges of the square lattice $\Lambda : N_v \times N_h$ (N_v columns and N_h rows) are arranged $N_v + 1$ types of spin variables. On horizontal edges (labelled by $\alpha = 1 \ldots N_h$) are arranged the spins σ taking spin projection $m_\alpha \in \{\pm 1/2\}$. On the columns (labelled by $j = 1 \ldots N_v$) the spin variables S_j can take any value $m_j \in \{-s_j, \ldots, +s_j\}$ of the s_jth representation of $su(2)$. The partition function is restricted to configurations for which an even number of spins are into (or out of) each lattice site (vertex); configurations in which the four spins are all in or all out are excluded (ice rule). The 'scattering' between spin states $(m_\alpha, m_j) \rightarrow (m'_\alpha, m'_j)$ of vertex (α, j) (see figure 1) have weights fixed by the universal matrix elements $R^{m_j, m'_j}_{m_\alpha, m'_\alpha}(\lambda - z_j)$ where λ is the spectral parameter. The quantities z_j (see also [13]) shift spectral parameters as inhomogeneities which we assume distributed only along the columns of Λ (figure 1).

The variables z_j take into account of disorder induced by the mixture of spin representations and/or by the actual distribution of impurities I_j in Λ: $z_j \equiv z(S_j) + z(I_j)$. We assume that both $z(S_j)$ and $z(I_j)$ enter the universal matrix (see equation (B.1)) in the same functional form. The disordered six-vertex model corresponds to the choice $z(S_j) = 0$, $z(I_j) \neq 0$. We impose periodic boundary conditions.

The transfer matrix $T(\lambda|\boldsymbol{z})$, where $\boldsymbol{z} := (z_1 \ldots z_{N_h})$, can be expressed in terms of rational R-matrices R_X, $X = \{\sigma, S\}$ (see equation (B.1)) which fulfil Yang–Baxter relations (see appendix B). This implies the integrability of the model: $[T(\lambda|\boldsymbol{z}), T(\mu|\boldsymbol{z})] = 0$.

The application of the transfer matrix to the Bethe vector $\Phi(\lambda_1 \ldots \lambda_N|\boldsymbol{z})$ reads

$$T(\lambda|\boldsymbol{z})\Phi(\lambda_1 \ldots \lambda_N|\boldsymbol{z}) = \Lambda(\lambda, \lambda_1 \ldots \lambda_N|\boldsymbol{z})\Phi(\lambda_1 \ldots \lambda_N|\boldsymbol{z}) - \sum_{\alpha=1}^{N} \frac{F_\alpha}{\lambda - \lambda_\alpha} \Phi_\alpha(\lambda_1 \ldots \lambda_{\alpha-1}, \lambda, \lambda_{\alpha+1} \ldots \lambda_N|\boldsymbol{z}) \tag{9}$$

(for the explicit form of the quantities $T(\lambda|\boldsymbol{z})$, $\Phi(\lambda_1 \ldots \lambda_N|\boldsymbol{z})$, $\Lambda(\lambda, \lambda_1 \ldots \lambda_N|\boldsymbol{z})$, F_α, and $\Phi_\alpha(\lambda_1 \ldots \lambda_{\alpha-1}, \lambda, \lambda_{\alpha+1} \ldots \lambda_N|\boldsymbol{z})$ in (9), see [15]). The condition for the diagonalization of T (and of the constants of motion generated by T) is that the spectral parameters are chosen to cancel the 'unwanted terms' (the second contribution to equation (9)) in the spectral problem (9); a sufficient condition is $F_\alpha = 0$ (algebraic BA equations). Such a condition has been termed as 'mass shell' constraint [15] imposed on equation (9). The OSBA spectral problem, instead, arises when the 'unwanted terms' are considered in equation (9); the spectral parameters obey a new set of equations called OSBA equations (see equation (13) below). The quasi-classical limit of the OSBA has remarkable properties. Namely, the solutions of the quasi-classical OSBA problem satisfy the Knizhnik–Zamolodchikov equations [21, 22] for the $su(2)$ CFT. In the following we shall see how the quasi-classical limit of the OSBA problem for the disordered vertex model is solved by spectral parameters fulfilling equation (4).

The quasi-classical limit of the vertex model is obtained through the expansion of $R_X(x;\eta)$ in powers of η ($R_X(x;0)$ is the identity).

Using the expressions for the monodromy, transfer, and universal matrix equations (B.1), (B.3), the quasi-classical limit of the OSBA equation (9) reads

$$\sum_{j=1}^{N_v} \frac{H_j}{\lambda - z_j}\phi = h\phi + \sum_{\alpha=1}^{N} \frac{f_\alpha}{\lambda - \lambda_\alpha}\phi_\alpha \tag{10}$$

up to order $\mathcal{O}(\eta^{N+2})$, where the explicit form of h and ϕ_α in (10) is given in [15]. By integrating equation (10) on a closed loop in the complex λ-plane encircling the pole $\lambda = z_j$ we obtain

$$H_j\phi = h_j\phi + \sum_{\alpha=1}^{N} \frac{f_\alpha}{z_j - \lambda_\alpha} S_j^+ \phi'_\alpha \tag{11}$$

$$h_j = \sum_{\substack{l=1\\ l\neq j}}^{N_v} \frac{s_l s_j}{z_l - z_j} - \sum_{\alpha=1}^{N} \frac{s_j}{\lambda_\alpha - z_j} \qquad j = 1 \ldots N_v \tag{12}$$

$$f_\alpha = \sum_{\substack{\beta=1\\ \alpha\neq\beta}}^{N} \frac{1}{\lambda_\alpha - \lambda_\beta} - \sum_{j=1}^{N_v} \frac{s_j}{\lambda_\alpha - z_j} \qquad \alpha = 1 \ldots N. \tag{13}$$

The Bethe vectors in the quasi-classical limit are

$$\begin{aligned} \phi &:= \prod_{\beta=1}^{N} S^+(\lambda_\beta, z)|\mathrm{vac}\rangle \\ \phi'_\alpha &:= \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} S^+(\lambda_\beta, z)|\mathrm{vac}\rangle. \end{aligned} \tag{14}$$

Here $|\mathrm{vac}\rangle = \otimes_{j=1}^{N_v}|s_j, -s_j\rangle$, where $S_j^-|s_j, -s_j\rangle = 0$, i.e. $|\mathrm{vac}\rangle$ is the highest weight vector in $\otimes_j su(2)_j$. The three operators $S^{\pm,z}(\lambda_\beta, z) := \sum_{j=1}^{N_v} S_j^{\pm,z}/(z_j - \lambda_\beta)$ generate higher-dimensional representations of the Gaudin algebra $\mathcal{G}[sl(2)]$, given by (see also equation (A.1))

$$\begin{aligned} [S^z(\lambda_\alpha, z), S^\pm(\lambda_\beta, z)] &= \pm\frac{S^\pm(\lambda_\alpha, z) - S^\pm(\lambda_\beta, z)}{\lambda_\beta - \lambda_\alpha} \\ [S^+(\lambda_\alpha, z), S^-(\lambda_\beta, z)] &= \frac{S^z(\lambda_\alpha, z) - S^z(\lambda_\beta, z)}{\lambda_\beta - \lambda_\alpha}. \end{aligned}$$

The 'mass shell' constraint $f_\alpha = 0$ corresponds to the diagonalization of the Gaudin model (see appendix A).

The solution of the spectral problem for the pairing model is recovered substituting

$$f_\alpha = \frac{1}{2}\left(\sum_{j=1}^{N_v} \frac{1-2s_j}{\lambda_\alpha - z_j} + \frac{1}{g}\right) \qquad \alpha = 1\ldots N \tag{15}$$

in the left-hand side of equation (13). In fact, the resulting equations coincide with equation (4). Substituting equation (15) in (11), and summing over index $j = 1\ldots N_v$ we obtain

$$\sum_{\alpha=1}^{N}\left(\sum_{j=1}^{N_v} \frac{4s_j - 1}{\lambda_\alpha - z_j} + \frac{1}{g}\right)\phi = 0 \tag{16}$$

where we have used the fact that $\sum_{j=1}^{N_v} h_j = -\sum_{\alpha=1}^{N}\sum_{j=1}^{N_v} s_j/(\lambda_\alpha - z_j)$. Equation (16) shows that the OSBA spectral problem is transformed in a spectral problem involving only diagonal matrix elements of suitably shifted (by f_α) transfer matrix of the vertex model (in the quasi-classical limit).

Since the limit $g \to \infty$ should correspond to the same result of $f_\alpha \to 0$ for generic s_j (compare with equation (A.4)), we impose that the distribution of spins S_j through the lattice fulfils the condition

$$\sum_{j=1}^{N_v} \frac{(1-2s_j)}{(\lambda_\alpha - z_j)} \equiv 0. \tag{17}$$

In this case equation (15) reduces to

$$f_\alpha = \frac{1}{2g} \qquad \alpha = 1, \ldots, N. \tag{18}$$

We choose $s_j = 1/2, \forall j$ in order to fulfil equation (17): the inhomogeneous vertex model becomes the disordered six-vertex model since $z(S_j) = 0$ and $z(I_j) \neq 0$. This implies that

$$H_j \equiv \Xi_j \qquad \text{and} \qquad \phi \equiv |N\rangle$$

where $N_v = \Omega$ (compare with equations (3) and (7)). Equation (16) can be transformed in the following eigenvalue equation:

$$\sum_{j=1}^{\Omega}\sum_{\alpha=1}^{N}\left(\frac{-\sigma_j^z}{2\varepsilon_j - e_\alpha} - \frac{1}{N}\Xi_j\right)\phi = \sum_{j=1}^{\Omega}\sum_{\alpha=1}^{N} \tau_{j,\alpha}\phi \tag{19}$$

$$\tau_{j,\alpha} := \left(\frac{1}{\Omega g} - \frac{1}{N} h_j\right) \tag{20}$$

where (4) (or (13), (18)), (12) and $\sum_{\beta=1}^{N} \sigma^z(e_\alpha, \varepsilon)\phi = 1/2\sum_{\beta=1}^{N}\sum_{j=1}^{\Omega} 1/(e_\alpha - 2\varepsilon_j)\phi$ have been used (the parameters in (12)–(20) are redefined as $z_j \leftrightarrow 2\varepsilon_j$, $\lambda_\alpha \leftrightarrow e_\alpha$). We point out that quantities in (20) are the eigenvalues of operators τ_j in equation (6) for generic Ω/N. At 'half-filling' $\Omega = 2N$ equation (20) reduces to

$$\tau_{j,\alpha} = \frac{1}{\Omega}\left(\frac{1}{g} - 2h_j\right) \qquad (j = 1, \ldots, \Omega). \tag{21}$$

Equations (20) and (21) coincide with those ones found by Sklyanin [10] and by Sierra [20].

The main result obtained in this paper is the connection between equations (11) and (19), (20) through (15). The OSBA problem for the disordered six-vertex model (which does not account for diagonalizing the transfer matrix of the vertex model) reveals the existence of a class of spectral problems (parametrized by f_α) which turns out to be diagonal on the quasi-classical Bethe vectors basis. For f_α fixed by (18) the diagonalization of the BCS model is obtained.

Furthermore, what we have discussed so far implies that the pairing Hamiltonian can be obtained from functionals of τ_j whose quasi-classical expansions have the following form:

$$\mathcal{T}(e|z) = \sum_{a=0}^{\infty} \eta^{2a} \mathrm{e}^{a-1} \left[\frac{1}{2g^2} + \tau(e) \right]^a \tag{22}$$

with

$$\tau(e) := \sum_{j=1}^{\Omega} \frac{\tau_j}{e - 2\varepsilon_j}. \tag{23}$$

We point out that $[\mathcal{T}(e|z), \mathcal{T}(e'|z)] = 0, \forall e, e'$ since quantities τ_j commute with each other. The residue in the poles $e = 2\varepsilon_j$ of the η^2 coefficient provides the integrals of motion τ_j. The residue of the η^4 coefficient reads (see equation (8))

$$\sum_{j,l=1}^{\Omega} \tau_j \tau_l + \frac{1}{g^2} \sum_{j=1}^{\Omega} \varepsilon_j \tau_j = \frac{1}{g^3} H. \tag{24}$$

4. Conclusions

We have established a novel connection between the disordered six-vertex model and the BCS model for generic g through the OSBA procedure. The BCS model is diagonalized by the quasi-classical limit of the OSBA equations of the disordered six-vertex model. Retaining certain off-diagonal terms of the transfer matrix of the vertex model corresponds to the diagonalization of the integrals of motion of the pairing model for finite g. The 'mass shell' condition (and then the diagonalization of the quasi-classical transfer matrix of the vertex model) reproduces the limit $g \to \infty$; the corresponding problem is the Gaudin spectral problem.

The integrals of motion of the BCS model coincide with the integrals found by Sklyanin [10] by considering a twist in the monodromy matrix of the vertex model (see equation (B.3)). The algebraic equations which diagonalize these integrals of motion via algebraic BA (namely via the mass shell BA procedure) coincide with Richardson's equations. This paper shows that the Sklyanin procedure produces the same results as the OSBA procedure applied to the *untwisted* monodromy matrix.

The existence of the relation between BCS model and quasi-classical vertex models, found in the present paper, is consistent with the correspondence between CFT and the BCS model recently found by Sierra [20].

Equations (4) were already conjectured by Gaudin (see (5.15) and (5.16) of [8]) as connected (through Jacobian of certain matrices) with the norms of the Bethe vectors, $\det(\partial \mathrm{f}_\alpha / \partial \mathrm{e}_\beta) \sim \|\phi\|$ (see [18,23]). In this work we have shown that the Jacobian is connected with OSBA of the vertex model. This might be useful to compute norms (and scalar products) and, then, to express the correlation functions of the BCS model as suitable determinants. This exact calculation is our major task in the future.

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Appendix A. The pairing model and the Gaudin spectral problem

In this appendix we discuss the connection between the pairing model and the Gaudin model.

The limit $g \to \infty$ of the constants of motion τ_j (6) coincides with Hamiltonians Ξ_j. Following equation (6), the spectrum of the pairing problem coincides with that of the Gaudin magnet [8]: $\Xi(u) := \sum_{j=1}^{\Omega} \Xi_j/(u - 2\varepsilon_j)$ (u is a complex parameter). The total energy is $h(u) := \sum_{j=1}^{\Omega} h_j/(u - 2\varepsilon_j)$ (h_j is fixed by equation (12)). The Bethe vectors of the Gaudin and the pairing problems coincide formally for any g since operators $(\sigma^{\pm}(u, \varepsilon), \sigma^z(u, \varepsilon))$ in (3) generate the Gaudin algebra $\mathcal{G}[sl(2)]$ in the lowest representation:

$$\begin{aligned} [\sigma^z(u, \varepsilon), \sigma^{\pm}(w, \varepsilon)] &= \pm \frac{\sigma^{\pm}(u, \varepsilon) - \sigma^{\pm}(w, \varepsilon)}{w - u} \\ [\sigma^{+}(u, \varepsilon), \sigma^{-}(w, \varepsilon)] &= \frac{\sigma^z(u, \varepsilon) - \sigma^z(w, \varepsilon)}{w - u} \end{aligned} \tag{A.1}$$

where $\sigma^z(u, \varepsilon) := \sum_{j=1}^{\Omega} \sigma_j^z/(2\varepsilon_j - u)$.

However, the spectral parameters entering the eigenvectors of the two models satisfy, for generic g, a different equation (compare with equation (4))

$$\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{2}{e_\beta - e_\alpha} - \sum_{j=1}^{\Omega} \frac{1}{2\varepsilon_j - e_\alpha} = 0. \tag{A.2}$$

We point out that equation (A.2) is the limit $g \to \infty$ of equations (4) for the pairing model. In this limit the two models have the same eigenvectors (see equations (3), (4) and (A.2)).

Thus the diagonalization of the Gaudin model is equivalent to the diagonalization of the BCS model for $g \to \infty$.

The limit of large g in equation (8) gives

$$\begin{aligned} H &\approx -g \sum_{j,l=1}^{\Omega} \frac{1}{\varepsilon_j - \varepsilon_l} [(\varepsilon_j + \varepsilon_l)\sigma_j^z\, \sigma_l^z + \varepsilon_j \sigma_j^+\, \sigma_l^-] \\ &\equiv -\frac{g}{2} \sum_{j,l=1}^{\Omega} \sigma_j^+\, \sigma_l^- \end{aligned} \tag{A.3}$$

which (consistently) reproduces the Hamiltonian (1) for large g.

The QISM was applied to Gaudin magnet for generic spin $\boldsymbol{S}_j$ in [8, 24]. In this case the Gaudin Hamiltonians are $H_l := \sum_{\substack{j=1 \\ j\neq l}}^{\Omega} \boldsymbol{S}_l \cdot \boldsymbol{S}_j/(\varepsilon_j - \varepsilon_l)$. The spectral parameters obey

$$\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{1}{e_\beta - e_\alpha} - \sum_{j=1}^{\Omega} \frac{s_j}{2\varepsilon_j - e_\alpha} = 0 \tag{A.4}$$

where $-s_j$ corresponds to the highest weight vector of $\boldsymbol{S}_j$.

Appendix B. Integrability of the inhomogeneous vertex model

In this appendix we discuss the QISM of the inhomogeneous vertex model together with its quasi-classical expansion.

The universal matrix of the model reads

$$R_X(\lambda - z; \eta) = \mathbb{1} \otimes \mathbb{1} + f(\lambda - z, \eta)\sigma \otimes X \tag{B.1}$$

where $f(x, \eta) := 2\eta/(\eta - 2x)$ depending on the arbitrary parameter $\eta \in \mathbb{R}$. The physical R-matrix of the model corresponds to taking $X \equiv S$ in (B.1); the auxiliary one corresponds to

$X \equiv \sigma$ and $z = 0$. Both these matrices fulfil homogeneous ($S \equiv \sigma$) Yang–Baxter relations; in addition they fulfil the inhomogeneous ($S \neq \sigma$) one [12]:

$$R_\sigma^{12}(\lambda - \mu)R_S^{23}(\lambda - z)R_S^{12}(\mu - z) = R_S^{23}(\mu - z)R_S^{12}(\lambda - z)R_\sigma^{23}(\lambda - \mu) \tag{B.2}$$

where $R^{12} \doteq R \otimes 1\!\!1$ and $R^{23} \doteq 1\!\!1 \otimes R$ act on the vector space $V_1 \otimes V_2 \otimes V_3$; R^{12} and R^{23} act as the identity in the vector spaces 3 and 1 respectively. Equation (B.2) is the sufficient condition for which the model can be solved by diagonalizing the column to column (which is 2×2 matrix in operators S_j) transfer matrix (instead of the row to row one which is $(2s_j + 1) \times (2s_j + 1)$) obtained through the trace in the two-dimensional horizontal vector space (which is spanned by spins along the rows of Λ) labelled by '(0)':

$$T(\lambda|\boldsymbol{z}) := \mathrm{tr}_{(0)}\, J(\lambda|\boldsymbol{z}). \tag{B.3}$$

Twisted monodromy matrix is obtained by $J(\lambda|\boldsymbol{z}) \rightarrow \mathrm{e}^{a\sigma_j^z} J(\lambda|\boldsymbol{z})$. The transfer matrices commute at different values of spectral parameters: $[T(\lambda|\boldsymbol{z}), T(\mu|\boldsymbol{z})] = 0$ (this proves the integrability of the model) since the monodromy matrix $J(\lambda|\boldsymbol{z}) := \prod_{j=N_v}^{1} R_{S_j}^{(0)}(\lambda - z_j)$ satisfies $R_\sigma(\lambda - \mu)[J(\lambda|\boldsymbol{z}) \otimes J(\mu|\boldsymbol{z})] = [J(\mu|\boldsymbol{z}) \otimes J(\lambda|\boldsymbol{z})]R_\sigma(\lambda - \mu)$ (induced by equation (B.2). The matrices $R_{S_j}^{(0)}$ and R_σ are

$$R_{S_j}^{(0)}(x, \eta) := \begin{pmatrix} 1\!\!1 + f(x,\eta)S_j^z & f(x,\eta)S_j^- \\ f(x,\eta)S_j^+ & 1\!\!1 - f(x,\eta)S_j^z \end{pmatrix} \tag{B.4}$$

$$R_\sigma(\lambda - \mu, \eta) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.5}$$

where $b(\lambda - \mu) := \eta/(\eta - \lambda - \mu)$, $c(\lambda - \mu) := \lambda - \mu/(\lambda - \mu - \eta)$ (note that $R_\sigma(\lambda - \mu, \eta)$ is z-independent).

In the quasi-classical limit, the system generates a hierarchy of integrable systems in the quasi-classical limit since

$$\sum_{a=b+c=0}^{\infty} [T_b(\lambda|\boldsymbol{z}), T_c(\mu|\boldsymbol{z})] = 0 \tag{B.6}$$

where we have used the η-expansion of the transfer matrix: $T(\lambda|\boldsymbol{z}) = \sum_{a=0}^{\infty} \eta^a T_a(\lambda|\boldsymbol{z})$ (the sum in equation (B.6) is meant on ordered partitions of a including $b \vee c = 0$). Up to order η^2, the transfer matrix reads

$$T(\lambda|\boldsymbol{z}) = 2 1\!\!1 + 2\eta^2 \sum_{j=1}^{N_v} \frac{H_j}{\lambda - z_j} \tag{B.7}$$

where the Hamiltonians H_j in equation (B.7) are

$$H_j = \sum_{\substack{l=1 \\ l \neq j}}^{N_v} \frac{\boldsymbol{S}_l \cdot \boldsymbol{S}_j}{z_j - z_l} \qquad (j = 1, \ldots, N_v). \tag{B.8}$$

The transfer matrix (B.7) coincides with the quasi-classical expansion of the twisted Gaudin model's transfer matrix (see formula (1.16) of [10]).

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